On Bounding the Performance of Group-Wise Multiuser Detectors

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***Abstract*—We develop a union bound on the error performance of group-wise symbol detectors in frequency-flat Rayleigh fading channels. The bound is applicable to detectors that search over a subspace of the total symbol space. It provides a means to investigate tradeoffs among system parameters such as group size versus error performance.**

***Index Terms*—Signal detection, multiuser detection, group detection, union bound.**

I. Introduction

THE detection of multiple co-channel signals is a key to spectrally efficient wireless communications such as multiple-input multiple-output systems with either a single or multiple users. Joint maximum likelihood (JML) [1] is the optimum detection technique for such systems. Its computational complexity increases exponentially with the number of co-channel signals making it infeasible for most practical systems. A union bound on the performance of JML detection is derived in [1]–[3].

A common approach to reduced-complexity approximations of JML is to reduce the symbol search space by forming signal groups, e.g. [4]–[6]. This comes at the cost of degradation in error performance. We develop a union bound on the error performance of such detectors. This allows us to evaluate error performance without extensive simulation. Moreover, the bound can be used to determine detector parameters such as group size and to investigate tradeoffs among system parameters.

II. Joint Maximum Likelihood Detection

We consider a M-antenna receiver with N independent co-channel signals impinging on each antenna. The signals are transmitted through a frequency-flat Rayleigh fading channel. The received signal vector $\mathbf{y} \in \mathbb{C}^M$ is

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}, \tag{1}$$

where $\mathbf{H} \in \mathbb{C}^{M \times N}$ denotes the channel matrix, $\mathbf{s} \in \mathcal{A}^N$ is the N-dimensional transmitted symbol vector with $\mathbf{s} = [s_1, s_2, \ldots, s_N]^T$ and $\mathbf{n} \in \mathbb{C}^M$ is the vector representing white Gaussian noise. We assume equiprobable transmit symbols $s_n \in \mathbf{s}$ in each interval drawn from an alphabet $\mathcal{A}$. The JML detector chooses the symbol vector

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s} \in \mathcal{S}} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2 \tag{2}$$

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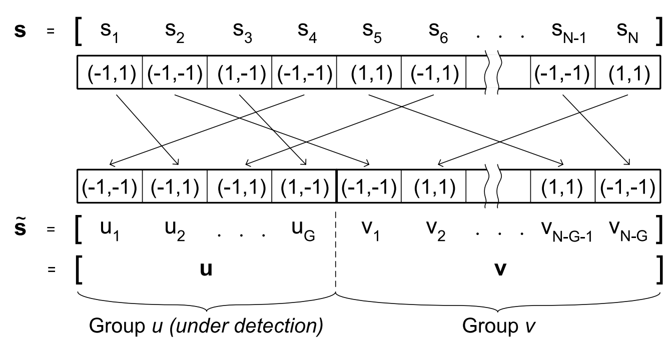

Fig. 1. Example of mapping the QPSK vector $\mathbf{s}$ into the vector $\tilde{\mathbf{s}}$. It contains a permutation of the symbols $s_n \in \mathbf{s}$ and is further divided into the subvectors $\mathbf{u}$ and $\mathbf{v}$. The symbol group u is the group being detected.

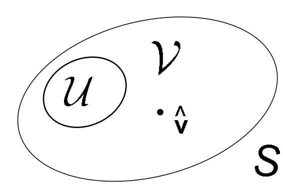

Fig. 2. Symbol space of a group-wise detector.

as the most likely estimate of the $\mathbf{s}$. Candidate symbols for $\mathbf{s}$ are drawn from the N-dimensional symbol space $\mathcal{S} = \mathcal{A}^N$.

III. Group Detection

We now describe group-wise detection. First, we map the transmit symbol vector $\mathbf{s}$ into a new vector $\tilde{\mathbf{s}} \in \mathcal{A}^N$. This allows us to specify arbitrary permutations of the symbols $s_n \in \mathbf{s}$ as illustrated in Fig. 1. The mapping operation is denoted

$$\mathbf{s}_{(N\times 1)} \longmapsto \tilde{\mathbf{s}}_{(N\times 1)}. \tag{3}$$

An arbitrary symbol group is then formed by expressing $\tilde{\mathbf{s}}$ in group form as

$$\tilde{\mathbf{s}} = [\mathbf{u}\; \mathbf{v}]^T, \tag{4}$$

where $\mathbf{u} \in \left(\mathcal{U} = \mathcal{A}^G\right)$ denotes a reduced-dimensionality group symbol vector and $\mathbf{v} \in \left(\mathcal{V} = \mathcal{A}^{N-G}\right)$ consists of all symbols outside the group. The group vector $\mathbf{u} = [u_1\; u_2\; \ldots\; u_G]^T$ contains G symbols and $\mathbf{v} = [v_1\; v_2\; \ldots\; v_{N-G}\;]^T$ contains $N-G$ symbols. Eqns. (3) and (4) define the division of the overall symbol space $\mathcal{S}$ into two subspaces $\mathcal{U}$ and $\mathcal{V}$. This is shown in Fig. 2. The group-wise symbol detector chooses candidate symbol vectors from the subspace $\mathcal{U}$.

The channel matrix $\mathbf{H}$ is also mapped into a new matrix $\widetilde{\mathbf{H}}$. It contains the column vectors of $\mathbf{H}$ in permutated order according to the mapping of (3). It is then split into two submatrices $\widetilde{\mathbf{H}}_u \in \mathbb{C}^{M\times G}$ and $\widetilde{\mathbf{H}}_v \in \mathbb{C}^{M\times(N-G)}$ such that

$$\widetilde{\mathbf{H}} = \left[\widetilde{\mathbf{H}}_u \; \widetilde{\mathbf{H}}_v\right]. \tag{5}$$

Using group notation, (1) can be written as

$$\mathbf{y} = \widetilde{\mathbf{H}}_u\mathbf{u} + \left[\widetilde{\mathbf{H}}_v\mathbf{v} + \mathbf{n}\right], \tag{6}$$

where the terms within the brackets are the undesired signal components. These are cancelled from the received vector $\mathbf{y}$ to form the signal vector for group u given by

$$\begin{aligned}\mathbf{y}_u &= \mathbf{y} - \widetilde{\mathbf{H}}_v\mathbf{v} \\ &= \widetilde{\mathbf{H}}_u\mathbf{u} + \mathbf{n}.\end{aligned} \tag{7}$$

The symbol vector $\mathbf{v}$ in (7) is often replaced by an estimate $\hat{\mathbf{v}}$, which is obtained in a separate estimation process[1]. A group-wise detector makes symbol decisions based on

$$\hat{\mathbf{u}} = \arg\min_{\mathbf{u}\in\mathcal{U}} \left\|\mathbf{y}_u - \widetilde{\mathbf{H}}_u\mathbf{u}\right\|^2. \tag{8}$$

The G group symbols $\hat{u}_n \in \hat{\mathbf{u}}$ are used to update a tentative symbol vector $\hat{\mathbf{s}}$. Symbol values for $\hat{\mathbf{v}}$ in (7) may also be drawn from $\hat{\mathbf{s}}$. The mapping, grouping and detection processes described by (3)-(8) are often employed iteratively in order to obtain improved estimates and to detect all N symbols $s_n \in \mathbf{s}$. After sufficient iterations, the group-wise detector outputs $\hat{\mathbf{s}}$.

IV. Union Bound

A union bound on the performance of a group-wise symbol detector is now derived. Our approach requires a different definition of the symbol subsets compared to the JML bound in [1]–[3]. We define the subset $\mathcal{T}(\hat{\mathbf{v}})$ as the set of N-dimensional symbol vectors $\{\mathbf{t}\}$ which have the symbol $u_n^{(k)}$ as their nth element ($n = 1, 2, \ldots, G$) *and* the vector $\hat{\mathbf{v}}$ as the estimate of $\mathbf{v}$. The superscript (k) denotes the kth symbol value in $\mathcal{A}$. There are $|\mathcal{A}|^{G-1}$ elements in $\{\mathbf{t}_j\} \in \mathcal{T}(\hat{\mathbf{v}})$. The complementary subset $\overline{\mathcal{T}}(\hat{\mathbf{v}})$ contains the remaining vectors $\{\mathbf{t}_i\}$ which do not have $u_n^{(k)}$ as their nth element. The subset $\overline{\mathcal{T}}(\hat{\mathbf{v}})$ contains $|\mathcal{A}|^N - |\mathcal{A}|^{G-1}$ elements $\{\mathbf{t}_i\}$. The Euclidean distance metrics for $\mathbf{t}_i$ and $\mathbf{t}_j$ are given by

$$\Lambda_i = \left\|\mathbf{y} - \widetilde{\mathbf{H}}\mathbf{t}_i\right\|^2 \tag{9}$$

and

$$\Lambda_j = \left\|\mathbf{y} - \widetilde{\mathbf{H}}\mathbf{t}_j\right\|^2, \tag{10}$$

respectively. The group-wise detector chooses the erroneous vector $\mathbf{t}_i$ over $\mathbf{t}_j$ if $\triangle_{ij} = \Lambda_j - \Lambda_i < 0$. A union bound on the symbol error probability $P_s(\hat{\mathbf{v}})$ of the nth signal ($n = 1, 2, \ldots, G$) is computed by evaluating the pairwise error probability (PEP) between all vectors $\{\mathbf{t}_j\} \in \mathcal{T}(\hat{\mathbf{v}})$ and $\{\mathbf{t}_i\} \in \overline{\mathcal{T}}(\hat{\mathbf{v}})$. This has the same form as the JML union bound (e.g. [3], eq. (5)) and takes into account the estimate $\hat{\mathbf{v}} \in \mathcal{V}$. For equiprobable transmit data vectors, the bound is obtained as

$$P_s(\hat{\mathbf{v}}) \le |\mathcal{A}|^{-G} \sum_n \sum_j \left(\sum_i P_{u_n^{(k)},\hat{\mathbf{v}},ij}\right), \quad n = 1, 2, \ldots G, \tag{11}$$

where $P_{u_n^{(k)},\hat{\mathbf{v}},ij}$ denotes the PEP between $\mathbf{t}_i$ and $\mathbf{t}_j$ given that $u_n^{(k)}$ is transmitted and that $\hat{\mathbf{v}}$ is the estimate of $\mathbf{v} \in \mathcal{V}$.

Errors in $\hat{\mathbf{v}}$ result in interference and increase the error probability of a group-wise detector. In contrast, if $\hat{\mathbf{v}}$ is correct, then (11) and the JML union bound [1]–[3] give the same result. Computation of $P_{u_n^{(k)},\hat{\mathbf{v}},ij}$ for the ideal case of perfect channel state information (CSI) at the receiver is shown in (12) and (13) at the bottom of the page. The derivation is analogous to [3] (eqns. (8), (9)). Here, $P_{u_n^{(k)},\hat{\mathbf{v}},ij}$ must consider the signal-to-interference-and-noise ratio (SINR), which for a given $u_n^{(k)}$ and $\hat{\mathbf{v}}$ is denoted $\Upsilon_{u_n^{(k)},\hat{\mathbf{v}}}$ and is defined as

$$\begin{aligned}\Upsilon_{u_n^{(k)},\hat{\mathbf{v}}} &= \frac{E_s}{I + N_0} = \frac{1}{\frac{I}{E_s} + \frac{N_0}{E_s}} \\ &= \frac{1}{\left(\Psi_{\hat{\mathbf{v}}}\right)^{-1} + \left(\Gamma_{u_n^{(k)}}\right)^{-1}},\end{aligned} \tag{14}$$

where E_s is the received symbol energy, N_0 is the noise power spectral density and I is the interference energy caused by symbol errors in $\hat{\mathbf{v}}$. We denote $\Psi_{\hat{\mathbf{v}}}$ as the signal-to-interference ratio (SIR) and $\Gamma_{u_n^{(k)}}$ as the signal-to-noise ratio (SNR).

If the estimation process for $\hat{\mathbf{v}}$ is error free, there is no interference ($I = 0$) and $\Upsilon_{u_n^{(k)},\hat{\mathbf{v}}} = \Gamma_{u_n^{(k)}}$. In contrast, if the SINR is dominated by the interference the noise power can be neglected and $\Upsilon_{u_n^{(k)},\hat{\mathbf{v}}} \approx \Psi_{\hat{\mathbf{v}}}$. A practical estimation process will in general contain errors so that the interference must be taken into account.

V. Results

We evaluate performance in terms of the symbol error rate (SER) per transmitted signal. We assume equal-energy QPSK signals transmitted through a frequency-flat Rayleigh fading channel. We further assume that the receiver has perfect CSI[2].

For the simulation of group-wise symbol detection we model the interference caused by estimation errors in $\hat{\mathbf{v}}$ as additive white Gaussian noise. Therefore, we first specify the SIR value and obtain the corresponding SINR using (14). The SINR is used to obtain a random variable with zero mean which is added to the received signal vector (instead of $\mathbf{n}$).

[1]See e.g. [4]–[6] for examples.

[2]The extension to the case of imperfect CSI is similar to that of [3].

$$P_{u_n^{(k)},\hat{\mathbf{v}},ij} = \frac{1}{\left(1 + r_{u_n^{(k)},\hat{\mathbf{v}},ij}\right)^{2M-1}} \sum_{m=0}^{M-1} \binom{2M-1}{m} \left(r_{u_n^{(k)},\hat{\mathbf{v}},ij}\right)^m \tag{12}$$

$$r_{u_n^{(k)},\hat{\mathbf{v}},ij} = a_{u_n^{(k)},\hat{\mathbf{v}},ij}\Upsilon_{u_n^{(k)},\hat{\mathbf{v}}} + \sqrt{\left(a_{u_n^{(k)},\hat{\mathbf{v}},ij}\Upsilon_{u_n^{(k)},\hat{\mathbf{v}}}\right)^2 + 2\left(a_{u_n^{(k)},\hat{\mathbf{v}},ij}\Upsilon_{u_n^{(k)},\hat{\mathbf{v}}}\right)} + 1, \quad a_{u_n^{(k)},\hat{\mathbf{v}},ij} = \frac{\|\mathbf{t}_i - \mathbf{t}_j\|^2}{2E_s} \tag{13}$$

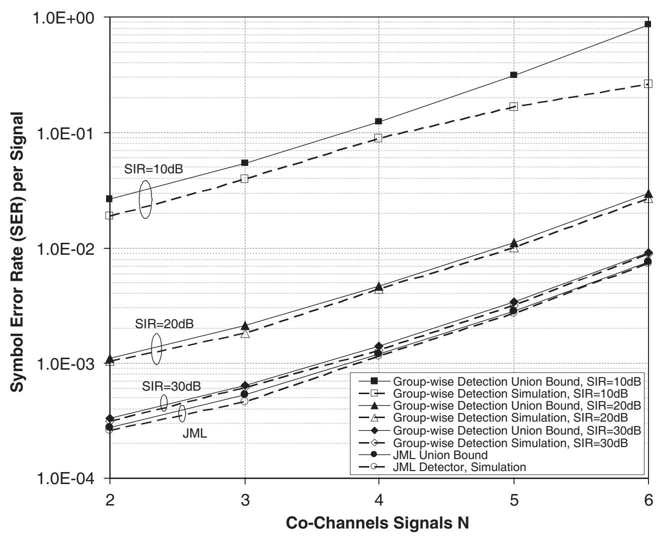

Fig. 3. SER of a group-wise symbol detector versus the number of co-channel signals N for a $M = 2$ antenna receiver and various SIR. The SNR is set to $\Gamma = 20dB$. Union bounds are shown by a solid line and simulation by a dashed line.

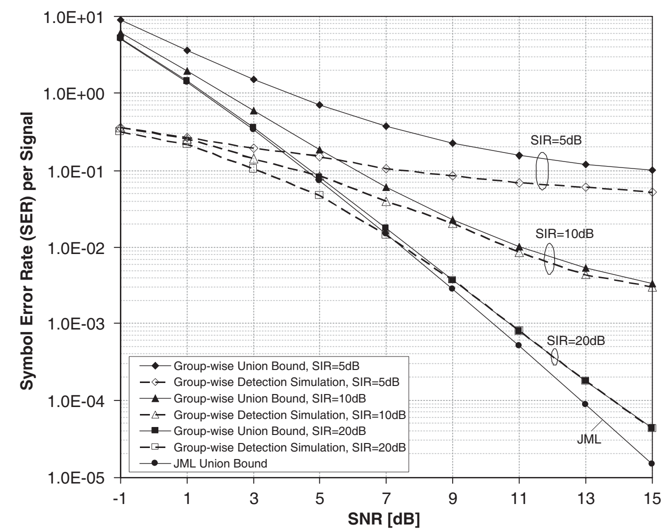

Fig. 4. SER versus SNR for an $M = 4$ antenna receiver with $N = 6$ co-channel signals and various SIR values. Solid lines indicate bounds and dashed lines show simulation results.

TABLE I
SIR AND SINR VALUES FOR SIMULATIONS IN FIG. 3. THE SNR IS SET TO $\Gamma = 20dB$.

SIR	$5dB$	$10dB$	$20dB$	$30dB$
SINR	$4.87dB$	$9.59dB$	$16.99dB$	$19.59dB$

We find the JML solution[3] of $\tilde{\mathbf{s}}$ which is used to obtain the estimate $\hat{\mathbf{v}} \in \mathcal{V}$. We then search over the group symbols $\mathbf{u} \in \mathcal{U}$ and use (8) to make symbol decisions.

Analytical performance results are obtained using the JML union bound of [3] (eq. (5)) and the new bound of (11) for group-wise detection. Computation of (11) requires knowledge of the estimate $\hat{\mathbf{v}}$ and the resulting interference in the form of the SIR $\Psi_{\hat{\mathbf{v}}}$. To keep the discussion general, we assume an arbitrary estimate $\hat{\mathbf{v}}$ and set the corresponding SIR $\Psi_{\hat{\mathbf{v}}}$ to a specified value. This differs from a practical detector, where it may be necessary to first measure the SIR of the estimation process for $\hat{\mathbf{v}}$ in order to compute (11). Table I shows the SIR and resulting SINR values used in simulations. The SNR at each antenna is assumed to be $\Gamma = 20dB$.

Fig. 3 depicts SER curves for a receiver with $M = 2$ antennas and different numbers of co-channel signals N. The SIR is set to $\Psi_{\hat{\mathbf{v}}} = 10, 20$ and $30dB$. In practice, the SIR often correlates with group size G and/ or the number of detector iterations. Larger values of G (smaller values of $N - G$) indicate higher SIR (and lower SER) due to more accurate estimation of $\mathbf{v} \in \mathcal{V}$. Some examples are shown in [4]–[6].

From Fig. 3 it is obvious that the JML detector achieves the best performance. The SER increases nearly linearly with the number of co-channel signals. Additional interference causes an upshift of the SER curves. The computed bounds are tight at high SINR but rather loose in the low SINR region. In [1]–[3], a similar dependency was observed for the JML union bound at different SNR.

Fig. 4 depicts the SER at different SNR for a receiver with $M = 4$ antennas and $N = 6$ co-channel signals. Results are shown for different SIR values. Again, the computed performance bounds are tight at high SINR (high SIR and high SNR). It can further be seen that for constant SIR an error floor occurs. The error floor is higher for small SIR values. This means that the overall performance of a group-wise detector is limited by the interference due to symbol errors in the vector $\hat{\mathbf{s}}$. For example, if the desired SER is $< 10^{-3}$ for SNR $\Gamma \geq 11dB$, the estimation process for $\hat{\mathbf{v}}$ must have an SIR of $\Psi_{\hat{\mathbf{v}}} \geq 20dB$. This may be used to specify system parameters such as group size and number of iterations of an iterative group-wise symbol detector.

[3] This represents an ideal estimation process for the symbols $\mathbf{v} \in \mathcal{V}$.

VI. CONCLUSIONS

We have presented a union bound on the error performance of group-wise symbol detectors in frequency-flat Rayleigh fading channels. The bound is tight in the high SINR region. Simulation results clearly show that the performance of a group-wise symbol detector is dominated by the SIR of the symbols outside the group under detection. If the SIR is known the proposed bound can be used to evaluate system parameters for group-wise multiuser detection.

REFERENCES

[1] S. J. Grant and J. K. Cavers, "Performance enhancement through joint detection of cochannel signals using diversity arrays," *IEEE Trans. Commun.,* vol. 46, pp. 1038-1049, Aug. 1998.

[2] Y. Kim, T. Kim, C. Kang, and D. Hong, "Union bound of maximum likelihood detection in downlink MIMO MC-CDMA systems," in *Proc. ICC,* vol. 6, June 2004.

[3] X. Zhu and R. D. Murch, "Performance analysis of maximum likelihood detection in a MIMO antenna system," *IEEE Trans. Wireless Commun.*, vol. 50, pp. 187-191, Feb. 2002.

[4] B. W. Zarikoff, J. K. Cavers, and S. Bavarian, "An iterative groupwise multiuser detector for overloaded MIMO applications," *IEEE Trans. Wireless Commun.,* vol. 6, pp. 443-447, Feb. 2007.

[5] M. Krause, D. P. Taylor, and P. A. Martin, "An iterative list-based multiuser detector for overloaded receivers in a Rayleigh fading channel," in *Proc. ICC,* Dresden, Germany, June 2009.

[6] M. Krause, "Signal detection for overloaded receivers," Ph.D. dissertation, University of Canterbury, Christchurch, New Zealand, 2009.